ON COUNTABLY COMPACT 0-SIMPLE TOPOLOGICAL INVERSE SEMIGROUPS

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Abstract. We describe the structure of 0-simple countably compact topological inverse semigroups and the structure of congruence-free countably compact topological inverse semigroups.

We follow the terminology of [3, 4, 8]. In this paper all topological spaces are Hausdorff. If S is a semigroup then we denote the subset of idempotents of S by $E(S)$. A topological space S that is algebraically a semigroup with a continuous semigroup operation is called a *topological semigroup*. A *topological inverse semigroup* is a topological semigroup S that is algebraically an inverse semigroup with continuous inversion. If Y is a subspace of a topological space X and $A{\subseteq}Y$, then we denote by $\operatorname{cl}_Y(A)$ the topological closure of A in Y.

The bicyclic semigroup $\mathscr{C}(p,q)$ is the semigroup with the identity 1 generated by two elements p and q, subject only to the condition $pq{=}1$. The bicyclic semigroup plays an important role in the algebraic theory of semigroups and in the theory of topological semigroups. For example, the well-known Andersen's result [1] states that a (0–) simple semigroup is completely (0–) simple if and only if it does not contain the bicyclic semigroup. The bicyclic semigroup admits only the discrete topology and a topological semigroup S can contain $\mathscr{C}(p,q)$ only as an open subset [7]. Neither stable nor Γ-compact topological semigroups can contain a copy of the bicyclic semigroup [2, 12].

Let S be a semigroup and I_λ a non-empty set of cardinality λ. We define the semigroup operation $'\cdot'$ on the set $B_\lambda(S){=}\ I_\lambda{\times}S^1{\times}I_\lambda{\cup}\{0\}$ as follows

$$(\alpha,a,\beta)\cdot(\gamma,b,\delta)=\begin{cases}(\alpha,ab,\delta), & \text{if } \beta=\gamma,\\ 0, & \text{if } \beta\neq\gamma,\end{cases}$$

and $(\alpha,a,\beta)\cdot 0{=}0\cdot(\alpha,a,\beta){=}0\cdot 0{=}0$, for $\alpha,\beta,\gamma,\delta{\in}I_\lambda$, and $a,b{\in}S^1$. The semigroup $B_\lambda(S)$ is called a *Brandt λ-extension* of the semigroup S [10]. Furthermore, if $A{\subseteq}S$ then we shall denote $A_{\alpha\beta}{=}\{(\alpha,s,\beta)\mid s{\in}A\}$ for $\alpha,\beta{\in}I_\lambda$. If a semigroup S is trivial (i.e. if S contains only one element), then $B_\lambda(S)$ is *the semigroup of $I_\lambda{\times}I_\lambda$-matrix units* [4], which we shall denote by B_λ. By Theorem 3.9 of [4], an inverse semigroup T is completely 0-simple if and only if T is isomorphic to a Brandt λ-extension $B_\lambda(G)$ of some group G and $\lambda{\geqslant}1$. We also note that if $\lambda{=}1$, then the semigroup $B_\lambda(S)$ is isomorphic to the semigroup S with adjoint zero. Gutik and Pavlyk [11] proved that any continuous homomorphism from the infinite topological semigroup of matrix units into a compact topological semigroup is annihilating, and hence the infinite topological semigroup of matrix units does not embed into a compact topological semigroup. They also showed that if a topological inverse semigroup S contains a semigroup of matrix units B_λ, then B_λ is a closed subsemigroup of S.

Suschkewitsch [17] proved that any finite semigroup S contains a minimal ideal K. He also showed that K is a completely simple semigroup and described the structure of finite simple semigroups. Rees [15] generalized the Suschkewitsch Theorem and showed that if

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a semigroup S contains a minimal ideal K then K is isomorphic to a Rees matrix semigroup $M[G; I, \Lambda, P]$ over a group G with a regular sandwich matrix P. He also proved that any completely 0-simple semigroup is isomorphic to a Rees matrix semigroup $M[G; I, \Lambda, P]$ over a 0-group G^0 with a regular sandwich matrix P. Wallace [18] proved the topological analogue of the Suschkewitsch-Rees Theorem for compact topological semigroups: *every compact topological semigroup contains a minimal ideal, which is topologically isomorphic to a topological paragroup.* Paalman-de-Miranda [14] proved that any 0-simple compact topological semigroup S is completely 0-simple, the zero of S is an isolated point in S and $S\setminus\{0\}$ is homeomorphic to the topological product $X\times G\times Y$, where X and Y are compact topological spaces and G is homeomorphic to the underlying space of a maximal subgroup of S, contained in $S\setminus\{0\}$. Owen [13] showed that if S a locally compact completely simple topological semigroup, then S has a structure similar to a compact simple topological semigroup. Owen also gave an example which shows that a similar statement does not hold for a locally compact completely 0-simple topological semigroup. Gutik and Pavlyk [11] proved that the subsemigroup of idempotents of a compact 0-simple topological inverse semigroup is finite, and hence the topological space of a compact 0-simple topological inverse semigroup is homeomorphic to a finite topological sum of compact topological group and a single point.

A Hausdorff topological space X is called *countably compact* if any open countable cover of X contains a finite subcover [8]. In this paper we shall prove that the bicyclic semigroup cannot be embedded into any countably compact topological inverse semigroup. We shall also describe the structure of 0-simple countably compact topological inverse semigroups and the structure of congruence-free countably compact topological inverse semigroups.

Theorem 1. *A countably compact topological inverse semigroup cannot contain the bicyclic semigroup. Therefore every (0-)simple countably compact topological inverse semigroup is (0-)completely simple.*

Proof. Let T be a countably compact topological inverse semigroup and suppose that T contains $\mathscr{C}(p,q)$ as a subsemigroup. Let $S = \operatorname{cl}_T(\mathscr{C}(p,q))$. Then by Theorem 3.10.4 of [8], S is a countably compact space and by Proposition II.2 of [7], S is a topological inverse semigroup. Thus by Corollary I.2 of [7], the semigroup $\mathscr{C}(p,q)$ is a discrete subspace of S and by Theorem I.3 of [7], $\mathscr{C}(p,q)$ is an open subspace of S and $S\setminus\mathscr{C}(p,q)$ is an ideal in S. Therefore any element of $\mathscr{C}(p,q)$ is an isolated point in the topological space S. We define the maps $\varphi\colon S\to E(S)$ and $\psi\colon S\to E(S)$ by the formulae $\varphi(x)=xx^{-1}$ and $\psi(x)=x^{-1}x$. Since $S\setminus\mathscr{C}(p,q)$ is an ideal of S, $A=\varphi^{-1}(\{1\})\cup\psi^{-1}(\{1\})\subseteq\mathscr{C}(p,q)$, and since the maps φ and ψ are continuous A is a clopen and hence countably compact infinite subset of S. But A is an open subspace of S whose elements are isolated points in S. A contradiction.

The second part of the theorem follows from Theorem 2.54 of [4]. $\square$

Let $\mathscr{S}$ be a class of topological semigroups. Let λ be a cardinal $\geqslant 1$, and $(S,\tau)\in\mathscr{S}$. Let τ_B be a topology on $B_\lambda(S)$ such that $(B_\lambda(S),\tau_B)\in\mathscr{S}$ and $\tau_B|_{(\alpha,S,\alpha)}=\tau$ for some $\alpha\in I_\lambda$. Then $(B_\lambda(S),\tau_B)$ is called a *topological Brandt λ-extension of* (S,τ) in $\mathscr{S}$ [10].

Let $\alpha,\beta,\gamma,\delta\in I_\lambda$ and A be a subspace of S. Since the restriction $\varphi^{\gamma\delta}_{\alpha\beta}\big|_{A_{\alpha\beta}}\colon A_{\alpha\beta}\to A_{\gamma\delta}$ of the map $\varphi^{\gamma\delta}_{\alpha\beta}\colon B_\lambda(S)\to B_\lambda(S)$ defined by the formula $\varphi^{\gamma\delta}_{\alpha\beta}(s)=(\gamma,1,\alpha)\cdot s\cdot(\beta,1,\delta)$ is a homeomorphism, we get the following:

Lemma 1. *Let $\lambda\geqslant 1$ and $B_\lambda(S)$ be a topological Brandt λ-extension of a topological semigroup S and A a subspace of S. Then the subspaces $A_{\alpha\beta}$ and $A_{\gamma\delta}$ in $B_\lambda(S)$ are homeomorphic for all $\alpha,\beta,\gamma,\delta\in I_\lambda$.*

Theorem 2. *Let S be a 0-simple countably compact topological inverse semigroup. Then there exist a nonempty finite set I_λ of cardinality λ and a countably compact topological group H such that S is topologically isomorphic to a topological Brandt λ-extension $B_\lambda(H)$ of H in the class of topological inverse semigroups. Moreover, S is homeomorphic to a finite topological sum of countable compact topological groups and a single point.*

Proof. By Theorem 1, the semigroup S is completely 0-simple. Now Theorem 3.9 of [4] implies that there exist a nonempty set I_λ of cardinality λ and a group G such that S is algebraically isomorphic to $B_\lambda(G)$. Therefore for any $\alpha{\in}I_\lambda$ the subset $G_{\alpha\alpha}$ is a subgroup of $B_\lambda(G)$ and since $B_\lambda(G)$ is a topological inverse semigroup, a topological subspace $G_{\alpha\alpha}$ of $B_\lambda(G)$ with the induced multiplication is a topological group. We fix $\alpha{\in}I_\lambda$ an put $H{=}G_{\alpha\alpha}$. Then the topological semigroup S is topologically isomorphic to a topological Brandt λ-extension $B_\lambda(H)$ of the topological group H.

Let e_H be the identity of H. Then the subsemigroup $B_\lambda(e_H){=}\{0\}\cup\{(\alpha,e_H,\beta)\mid\alpha,\beta{\in}I_\lambda\}$ of $B_\lambda(H)$ is algebraically isomorphic to the semigroup of matrix units B_λ. By Theorem 14 [11], $B_\lambda(e_H)$ is a closed subsemigroup of $B_\lambda(H)$ and hence by Theorem 3.10.4 of [8], $B_\lambda(e_H)$ is a countably compact topological space. Therefore Theorem 6 of [11] implies that $B_\lambda(e_H)$ is a finite discrete subsemigroup of $B_\lambda(H)$ and hence the set I_λ is finite.

We define the maps $\varphi\colon B_\lambda(H){\to}B_\lambda(e_H)$ and $\psi\colon B_\lambda(H){\to}B_\lambda(e_H)$ by the formulae $\varphi(x)=xx^{-1}$ and $\psi(x){=}x^{-1}x$. Since $B_\lambda(H)$ is a topological inverse semigroup the maps φ and ψ continuous and hence by Lemma 4 of [11], the set $H_{\alpha\beta}{=}\varphi^{-1}((\alpha,e_H,\beta)){\cap}\varphi^{-1}((\alpha,e_H,\beta))$ is clopen in $B_\lambda(H)$. By Lemma 1, the subspaces $H_{\alpha\beta}$ and $H_{\gamma\delta}$ are homeomorphic for any $\alpha,\beta,\gamma,\delta{\in}I_\lambda$, and hence all of them are homeomorphic to the topological group H. □

A Tychonoff topological space X is called *pseudocompact* if every continuous real-valued function on X is bounded. Since the topological space of T_0-topological group is Tychonoff and any topological sum of Tychonoff spaces is a Tychonoff space, Theorem 3.10.20 of [8] implies:

Corollary 1. *The topological space of a* 0*-simple countably compact topological inverse semigroup is Tychonoff and hence pseudocompact.*

Let X be a topological space. The pair (Y,c), where Y is a compactum and $c\colon X{\to}X$ is a homeomorphic embedding of X into Y, such that $\mathrm{cl}_Y\, c(X){=}Y$, is called a *compactification* of the space X. Define the ordering $\preccurlyeq$ on the family $\mathcal{C}(X)$ of all compactifications of a topological space X as follows: $c_2(X){\preccurlyeq}c_1(X)$ if and only if there exists a continuous map $f\colon c_1(X){\to}c_2(X)$ such that $fc_1{=}c_2$. The greatest element of the family $\mathcal{C}(X)$ with respect to the ordering $\preccurlyeq$ is called the *Stone-Čech compactification* of the space X and it is denoted by βX. Comfort and Ross [6] proved that the Stone-Čech compactification of a pseudocompact topological group is a topological group. The next theorem is an analogue of the Comfort–Ross Theorem:

Theorem 3. *Let S be a* 0*-simple countable compact topological inverse semigroup. Then the Stone-Čech compactification of S admits a structure of* 0*-simple topological inverse semigroup with respect to which the inclusion mapping of S into βS is a topological isomorphism.*

Proof. By Theorem 2, S is topologically isomorphic to a Brandt λ-extension of some topological group H in the class of topological inverse semigroups and $\lambda{<}\omega$. Now by Lemma 1, the subspaces $H_{\alpha\beta}$ and $H_{\gamma\delta}$ are homeomorphic in $B_\lambda(H)$, for any $\alpha,\beta,\gamma,\delta{\in}I_\lambda$. Since a maximal subgroup in S is closed we have that $H_{\alpha\beta}$ is a clopen subset of $B_\lambda(H)$, for every $\alpha,\beta{\in}I_\lambda$. By Corollary 1, the topological space $B_\lambda(H)$ is pseudocompact. Since any clopen subspace of a pseudocompact topological space is pseudocompact (see [5]) the subspace $H_{\alpha\beta}$ is pseudocompact, for every $\alpha,\beta{\in}I_\lambda$. Obviously, the topological space $B_\lambda(H)\backslash\{0\}$ is homeomorphic to $H{\times}I_\lambda{\times}I_\lambda$. Since the topological space $I_\lambda{\times}I_\lambda$ is finite and hence compact, by Corollary 3.10.27 of [8], the space $B_\lambda(H)\backslash\{0\}$ is pseudocompact. Now by Theorem 1 of [9], we have $\beta(H{\times}I_\lambda{\times}I_\lambda){=}\beta H{\times}\beta I_\lambda{\times}\beta I_\lambda{=}\beta H{\times}I_\lambda{\times}I_\lambda$ and therefore $\beta(B_\lambda(H)){=}B_\lambda(\beta H)$. □

Corollary 2. *Every* 0*-simple countable compact topological inverse semigroup is a dense subsemigroup of a* 0*-simple compact topological inverse semigroup.*

If S is completely simple inverse semigroup then the semigroup S with joined zero S^0 is completely 0-simple and hence by Theorem 3.9 of [4], the semigroup S^0 is isomorphic to a Brandt λ-extension $B_\lambda(G)$ of some group G. Therefore any nonzero idempotent of S^0 is

primitive. Let e and f are nonzero idempotents of S^0. Since S is an inverse subsemigroup of S^0 we have $ef{=}fe{\leqslant}e$ and $ef{=}fe{\leqslant}f$, and hence $e{=}ef{=}f$. Thus, the inverse semigroup S contains the unique idempotent and hence it is a group. Therefore a completely simple inverse semigroup is a group and Theorem 1 implies that *every simple countable compact topological inverse semigroup is a topological group.*

A semigroup S is called *congruence-free* if it has only two congruences: the identity relation and the universal relation [16].

Theorem 4. *Let S be a congruence-free countably compact topological inverse semigroup with zero. Then S is isomorphic to a finite semigroup of matrix units.*

Proof. Suppose not. Since the semigroup S contains a zero by Theorem 2, S is topologically isomorphic to a topological Brandt λ-extension $B_\lambda(H)$ of a pseudocompact topological group H in the class of topological inverse semigroups and $\lambda{<}\omega$. Suppose that the group H is not trivial. Then we define a map $h\colon B_\lambda(H){\to}B_\lambda$ by the formulae $h((\alpha,g,\beta)){=}(\alpha,\beta)$ and $h(0){=}0$. Since $h((\alpha,g,\beta)(\gamma,s,\delta)){=}\, h((\alpha,gs,\delta)){=}(\alpha,\delta) = (\alpha,\beta)(\gamma,\delta){=}\, h((\alpha,g,\beta))h((\gamma,s,\delta))$ for $\beta{=}\gamma$ and $h((\alpha,g,\beta)(\gamma,s,\delta)) = h(0){=}0{=}(\alpha,\beta)(\gamma,\delta) = h((\alpha,g,\beta))h((\gamma,s,\delta))$ for $\beta\neq\gamma$, the map h is a homomorphism. This contradicts the assumption that S is a congruence-free semigroup. $\square$

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